A CHARACTERIZATION OF WEAKLY REGULAR LINEAR FUNCTIONALS

by

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Abstract


A linear functional is said to be weakly-regular if it is not a finite sum of Dirac masses and their derivatives. In this paper, we consider the first-order linear differential equations \((Eu)' + Fu = 0\) where \(u\) is a non-zero linear functional and \((E, F)\) is a pair of polynomials, with \(E\) monic. The aim of this work is to give weak-regularity conditions on \(u\). Under certain admissibility conditions of the pair \((E, F)\), the weak-regularity of \(u\) leads to its regularity. Some examples are analyzed.

Key words: First-order linear differential equations, weak-regular and regular functionals, weak-semiclassical and semi-classical functionals.

Resumen

Un funcional lineal se dice débilmente regular si no es la suma finita de masas de Dirac y sus derivadas. En este trabajo consideramos las ecuaciones diferenciales lineales de primer orden \((Eu)' + Fu = 0\), donde \(u\) es un funcional lineal no nulo y \((E, F)\) es una pareja de polinomios, con \(E\) monico. El propósito de este trabajo es dar condiciones de regularidad débil sobre \(u\). Bajo ciertas condiciones de admisibilidad de la pareja \((E, F)\), la regularidad débil de \(u\) conduce a su regularidad. Se analizan algunos ejemplos.

Palabras clave: Ecuaciones diferenciales lineales de primer orden, funcionales regulares y débilmente regulares, funcionales semiclassicos, funcionales débilmente semiclassicos.

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Introduction

Let $u$ be a non-zero linear functional satisfying the following first-order linear differential equation

$$(Eu)'+Fu=0 \quad (*)$$

where $E$ and $F$ are non-zero polynomials, with $E$ a monic polynomial.

When the linear functional $u$ is regular then it is said to be semiclassical [6, 7].

Notice that a linear functional $u$ is said to be regular [4, 7] if there exists a monic polynomial sequence (MPS) $\{B_n\}_{n \geq 0}$ where $\text{deg } B_n = n$, $n \geq 0$, such that

$$\langle u, B_n B_m \rangle = r_n \delta_{n,m}, \quad n, \ m \geq 0, \quad r_n \neq 0, \ n \geq 0.$$ 

Besides regular functionals, the equation $(*)$ can have as solutions linear functionals defined as a finite sum of Dirac masses and their derivatives. In such a case there exists a non-zero polynomial $\phi$ such that $\phi u = 0$. More precisely, if

$$u = \sum_{i=1}^{t} \sum_{j=0}^{k_i-1} M_{i,j} \delta^{(j)}(x-x_i),$$

then $\phi(x) = \prod_{i=1}^{t} (x-x_i)^{k_i}$. Obviously, such linear functionals $u$ are not regular. For this reason, we introduce in a natural way the concept of weak-regularity linear functional $u$ as follows.

A non-zero linear functional $u$ is said to be weakly-regular if for a polynomial $\phi$ such that $\phi u = 0$, then $\phi = 0$. Regular linear functionals are weakly-regular (in general the converse is not true, see Remark 1.6).

In this paper, we are dealing with weak-semiclassical linear functionals, i.e., when the linear functional $u$ satisfying $(*)$ is weakly-regular. The aim of our contribution is to give essentially a necessary and sufficient condition for the weak-regularity of a non-zero linear functional $u$ satisfying $(*)$.

The paper is organized as follows. In Section 1, we introduce the basic notations and tools that will be used throughout the paper. Next, we define the weak-regularity of a linear functional and we analyze some properties like the stability by the shifting perturbation of the linear functional as well as the left multiplication of the linear functional by a polynomial. We conclude this section introducing the notion of admissible pair of polynomials. In section 2, our main results are proved. We obtain a necessary and sufficient condition in order to a non-zero linear functional $u$ satisfying a first-order linear differential equation $(Eu)'+Fu=0$ be weakly-regular. This yields the definition of weak-semiclassical functional. In section 3, we prove (Proposition 3.2) that the classical functionals are the only weakly-regular functionals satisfying $(Eu)'+Fu=0$, where $E$ and $F$ are two polynomials, $E$ monic, $\text{deg } E \leq 2$, $\text{deg } F = 1$, and the pair $(E, F)$ is admissible. This result generalizes one by Geronimus on classical functionals, see [5]. In section 4, the results of section 3 are used to characterize semiclassical polynomial sequences, which are orthogonal with respect to regular functionals $u$ given by $Au = \lambda B v$, where $A$ and $B$ are two monic polynomials, $\lambda \in \mathbb{C}^*$, and $v$ is a classical linear functional.

1. Definitions and background

Let $\mathbb{P}$ be the linear space of complex polynomials in one variable and $\mathbb{P}'$ its topological dual space. We denote by $(u, f)$ the action of $u \in \mathbb{P}'$ on $f \in \mathbb{P}$ and by $(u)_{c,n} := (u, (x-c)^n)$, $n \geq 0$, the moments of $u$ with respect to the sequence $\{(x-c)^n\}_{n \geq 0}$. In particular, if $c = 0$, then we will denote $(u)_{n} := (u)_{0,n}$, $n \geq 0$.

We define the following operations in $\mathbb{P}'$. For any linear functional $u$, any polynomial $h$, and any $c \in \mathbb{C}$, let $D u = u'$, $hu$, $(x-c)^{-1}u$, and $\sigma(u)$ be the linear functionals defined by duality

$$\langle u', f \rangle := -\langle u, f' \rangle, \quad f \in \mathbb{P},$$

$$\langle h u, f \rangle := \langle u, h f \rangle, \quad f \in \mathbb{P},$$

$$\langle (x-c)^{-1} u, f \rangle := \langle u, \theta_c(f) \rangle, \quad f \in \mathbb{P},$$

$$\langle \sigma(u), f \rangle := \langle u, \sigma(f) \rangle, \quad f \in \mathbb{P},$$

where $\theta_c(f)(x) = \frac{f(x)-c}{x-c}$ and $\sigma(f)(x) = f(x^2)$. Notice that

$$f(x) \sigma(u) = \sigma(f(x^2)u), \quad f \in \mathbb{P}.$$  \hspace{1cm} (1.1)

Let $\{B_n\}_{n \geq 0}$ be a monic polynomial sequence (MPS), $\text{deg } B_n = n$, $n \geq 0$, and $\{u_n\}_{n \geq 0}$ its dual sequence, $u_n \in \mathbb{P}'$, $n \geq 0$, defined by $(u_n, B_m) := \delta_{n,m}$, $n \geq 0$, where $\delta_{n,m}$ is the Kronecker symbol.

The linear functional $u_0$ is said to be the canonical functional associated with the MPS $\{B_n\}_{n \geq 0}$.

We remind the following results [2, 4, 7].

Lemma 1.1. For any $u \in \mathbb{P}'$ and any integer $m \geq 1$, the following statements are equivalent

i) $\langle u, B_{m-1} \rangle \neq 0$, $\langle u, B_n \rangle = 0$, $n \geq m$.

ii) There exist $\lambda_v \in \mathbb{C}$, $0 \leq v \leq m-1$, $\lambda_{m-1} \neq 0$ such that $u = \sum_{v=0}^{m-1} \lambda_v u_v$. 

Let \( \phi \) be a polynomial such that \( \phi u = 0 \). We can always write \( \phi(x) = A(x^2) + xB(x^2) \) where \( A \) and \( B \) are polynomials. For every \( p \in \mathbb{P} \), one has
\[
0 = \langle \phi u, p(x^2) \rangle = \langle u, \phi(x)p(x^2) \rangle = \langle u, A(x^2)p(x^2) \rangle = \langle u, A(x)p(x) \rangle = \langle A\sigma(u), p(x) \rangle.
\]
Then, \( A\sigma(u) = 0 \). So, from the weak-regularity of \( \sigma(u) \) we deduce \( A = 0 \). On the other hand,
\[
0 = \langle \phi u, xp(x^2) \rangle = \langle u, x\phi(x)p(x^2) \rangle = \langle u, x^2B(x^2)p(x^2) \rangle = \langle u, xB(x)p(x) \rangle = \langle xB\sigma(u), p(x) \rangle.
\]
Then, \( xB\sigma(u) = 0 \). So, \( B = 0 \) taking into account the weak-regularity of \( \sigma(u) \). Thus \( \phi = 0 \).

Notice that if \( u \) is a symmetric regular linear functional then \( xu \) is a weakly-regular functional that is not regular.

1.2. Admissibility pair of polynomials. Let \( (E,F) \) be a pair of polynomials, where \( E \) monic, \( \deg E = t \), \( \deg F = p \geq 1 \), and \( s(E,F) := \max(t - 2, p - 1) \). Notice that \( (s(E,F) \geq 0 \), because \( \deg F \geq 1 \). For this pair of polynomials, we introduce

- the admissibility coefficients

\[
\Delta_n(E,F) = nE^{(s+2)}(0) - (s+2)F^{(s+1)}(0), \quad n \geq 0; \quad (1.6)
\]

- the sequence of polynomials

\[
F_m = F - (m-1)E', \quad m \geq 1. \quad (1.7)
\]

Definition 1.8. The pair \( (E,F) \) is said to be admissible when its admissibility coefficients satisfy
\[
\Delta_n(E,F) \neq 0, \quad n \geq 0. \quad (1.8)
\]

From an admissible pair of polynomials, we can deduce other admissible pairs. Indeed, we have the following result.

Lemma 1.9. When \( (E,F) \) is admissible, then for each integer \( m \geq 1 \), we have
\[\begin{align*}
&\text{i) } \deg F_m \geq 1, \text{ where } F_m = F - (m - 1)E', \quad m \geq 1. \\
&\text{ii) } s(E,F_m) = s(E,F) := s. \\
&\text{iii) } The \text{ pair } (E,F_m) \text{ is admissible and} \\
&\Delta_n(E,F_m) = \Delta_{n+(m-1)(s+2)}(E,F), \quad n \geq 0.
\end{align*}\]

Proof. Assume there exists an integer, \( m \geq 1 \), such that \( F_m \) is a constant polynomial. Since \( \deg F \geq 1 \), then \( m \geq 2 \). In this case, \( s = t - 2 = p - 1 \) and the coefficient of \( x^p \) in \( F_m \) is \( (p!)^{-1}F^{(p)}(0) - (m - 1) = 0 \). Then \( \Delta_{(m-1)}(E,F) = 0 \), and this contradicts the admissibility condition of the pair \( (E,F) \). Hence, i) holds. The admissibility condition of the pair \( (E,F) \) yields
\[
\deg F_m = \max(p, t - 1) = s + 1, \quad m \geq 1. \quad (1.9)
\]
Thus,
\[
s(E,F_m) = \max(t - 2, s) = s.
\]
Hence, ii) holds.

From i), ii), and (1.6), one has
\[
\Delta_n(E,F_m) = nE^{(s+2)}(0) - (s+2)F_m^{(s+1)}(0)
\]
\[
= (n + (m - 1)(s + 2))E^{(s+2)}(0)
\]
\[
- (s + 2)F^{(s+1)}(0)
\]
\[
= \Delta_{(m-1)(s+2)}(E,F), \quad n \geq 0.
\]
Thus, the admissibility condition of the pair \( (E,F_m) \) follows from the admissibility of the pair \( (E,F) \). Hence, iii) holds.

For each fixed \( (a,b) \in \mathbb{C}^* \times \mathbb{C} \), we can consider the shifted pair \( (\tilde{E},\tilde{F}) \) given by
\[
\tilde{E}(x) := a^{-t}E(ax + b) \quad ; \quad \tilde{F}(x) := a^{1-t}F(ax + b).
\]
Let denote \( \tilde{s} = \max(\tilde{t} - 2, \tilde{p} - 1) \), where \( \tilde{t} = \deg(\tilde{E}) \) and \( \tilde{p} = \deg(\tilde{F}) \). Thus
\[
\tilde{t} = t, \quad \tilde{p} = p, \quad \tilde{s} = s. \quad (1.11)
\]
As a consequence the following result holds.

Lemma 1.10. If \( (E,F) \) is admissible, then \( (\tilde{E},\tilde{F}) \) is also admissible. Furthermore,
\[
\Delta_n(\tilde{E},\tilde{F}) = a^{s+2-t}\Delta_n(E,F), \quad n \geq 0. \quad (1.12)
\]

Proof. If \( (a,b) \in \mathbb{C}^* \times \mathbb{C} \), then
\[
\Delta_n(\tilde{E},\tilde{F}) = n\tilde{E}^{(s+2)}(0) - (s + 2)\tilde{F}^{(s+1)}(0)
\]
\[
= a^{s+2-t}\left( nE^{(s+2)}(0) - (s + 2)F^{(s+1)}(0) \right)
\]
\[
= a^{s+2-t}\Delta_n(E,F), \quad n \geq 0.
\]
Hence, i) follows.

When the pair of polynomials \( (E,F) \) is admissible and \( \deg E \geq 1 \), we deduce the following results.
Lemma 1.11. Let \((E, F)\) be an admissible pair of polynomials, where \(\deg E \geq 1\), and \(c\) is a zero of \(E\). Then for each integer \(m \geq 1\), we get

i) \(\deg \hat{F}_m \geq 1\), where \(\hat{F}_m = F - (m - 1)\theta_c(E)\).

ii) \(s(E, \hat{F}_m) = s(E, F) := s\).

iii) The pair \((E, \hat{F}_m)\) is admissible, and
\[
\Delta_n(E, \hat{F}_m) = \Delta_{n+m-1}(E, F), \quad n \geq 0.
\]

Proof. Assume there exists an integer \(m \geq 1\) such that \(\hat{F}_m\) is a constant polynomial. Since \(\deg F \geq 1\), then \(m \geq 2\). In this case, \(s = t - 2 = p - 1\) and the coefficient of \(x^p\) in \(\hat{F}_m\) is \((pl)^{-1} F(p) - (m - 1) = 0\). Then \(\Delta_{m-1}(E, F) = 0\), and this contradicts the admissibility condition of the pair \((E, F)\). Hence, i) holds. The admissibility condition of the pair \((E, F')\) means that

\[
\deg(\hat{F}_m) = \max(p, t - 1) = s + 1, \quad m \geq 1. \quad (1.13)
\]

Thus,

\[
s(E, \hat{F}_m) = \max(t - 2, s) = s, \quad m \geq 1.
\]

Hence, ii) holds. For \(m \geq 1\), from i), ii), and (1.6) one has

\[
\Delta_n(E, \hat{F}_m) = nE^{(s+2)}(0) - (s + 2)\hat{F}_m^{(s+1)}(0).
\]

Since \(t \leq s + 2\), it follows that

\[
E^{(s+2)} = (s + 2)(\theta_c(E))^{(s+1)}.
\]

Thus,

\[
\Delta_n(E, \hat{F}_m) = (n + m - 1)E^{(s+2)}(0) - (s + 2)F^{(s+1)}(0)
\]

\[
= \Delta_{n+m-1}(E, F), \quad n \geq 0.
\]

Hence, iii) holds. \(\square\)

For the sequel, we need the following results.

Lemma 1.12. Let \((E, F)\) be a pair of non-zero polynomials, where \(E\) monic, \(\deg E \geq 1\). If \(E\) and \(F\) are coprime, then

i) There exists an integer \(\mu \geq 1\) such that \(E\) and \(F_m = F - (m - 1)E'\) are coprime, \(m \geq \mu\).

ii) For each zero \(c\) of \(E\), there exists an integer \(\vartheta \geq 1\) such that \(E\) and \(\hat{F}_m = F - (m - 1)\theta_c(E)\) are coprime, \(m \geq \vartheta\).

Proof. Assume that for each integer \(\mu \geq 1\), there exists an integer \(m_\mu \geq \mu\) such that \(E\) and \(F - (m_\mu - 1)E'\) have a common zero. Then there is a zero \(c\) of \(E\) and two different integers \(m_{\nu \vartheta} \geq 1, \nu = 1, 2\) respectively, such that \((F - (m_{\nu \vartheta} - 1)E')(c) = 0, \nu = 1, 2\). This yields \(F(c) = 0\), that contradicts the fact that \(E\) and \(F\) are coprime. Hence, i) holds. Let \(c\) be a zero of \(E\). Two cases must be analyzed.

Case 1. Let assume \(c\) is a simple zero of \(E\). Suppose that for each integer \(\vartheta \geq 1\), there exists an integer \(m_\vartheta \geq \vartheta\) such that \(E\) and \(F - (m_\vartheta - 1)\theta_c(E)\) have a common zero. Then it will exist a zero \(c\) of \(E\) and two different integers \(m_{\nu \vartheta} \geq 1, \nu = 1, 2\) such that \((F - (m_{\nu \vartheta} - 1)\theta_c(E))(c) = 0, \nu = 1, 2\). This leads to \(F(c) = 0\), in contradiction with the fact that \(E\) and \(F\) are coprime.

Case 2. \(c\) is a zero of \(E\) with multiplicity at least two. For every zero \(\xi\) of \(E\), we have \(\hat{F}_m(\xi) = F(\xi) - (m - 1)\theta_c(E)(\xi) = F(\xi) \neq 0, m \geq 1\). Hence, ii) holds.

As a consequence, for a pair of non-zero polynomials \((E, F)\), where \(E\) is a monic polynomial, \(\deg E \geq 1\), and where \(E\) and \(F\) are coprime, we can associate the integer

\[
\mu(E, F) := \min\{k \geq 1 : E\text{ and } F_m\text{ are coprime}, m \geq k\}.
\]

(1.14)

2. Main Results

Let \((E, F)\) be a pair of polynomials, with \(E\) monic, \(\deg E = t\), \(\deg F = p \in \mathbb{N} \cup \{-\infty\}\), and \(s := s(E, F)\). Consider the functional equation

\[
(Eu)' + Fu = 0, \quad u \in \mathbb{P}^*.
\]

(2.1)

Lemma 2.1. Let \(u \in \mathbb{P}^*\) a solution of (2.1). When the pair \((E, F)\) is admissible and the \((s+1)\text{- first moments} \quad (u)_0, ..., (u)_s\) are fixed, then \(u\) is unique.

Proof. The admissibility condition of the pair \((E, F)\) requires that \(p \geq 1\). Then, \(s \geq 0\). The functional equation (2.1) is equivalent to the following recurrence relation for the corresponding moments

\[
\sum_{\nu=0}^{s+2} \frac{\Delta_{n+\nu}(E, F)}{\nu!} (u)_{n+\nu} = 0, \quad n \geq 0,
\]

(2.2)

where \(\Delta_{n+\nu}(E, F) := nE^{(\nu)}(0) - \nu F^{(\nu-1)}(0), 0 \leq \nu \leq s + 2\). Suppose that \(v \in \mathbb{P}^*\) is other solution of (2.1). Then, the linear functional \(w = v - u\) satisfies

\[
\sum_{\nu=0}^{s+2} \frac{\Delta_{n+\nu}(E, F)}{\nu!} (w)_{n+\nu} = 0, \quad n \geq 0.
\]
where $F_0 = F - (\nu - 1)E'$. Notice that $A'E - AF_0 \neq 0$. Otherwise, $A'E = AF_0$. Since $E$ and $F_0$ are coprime $E$ divides $A$, a contradiction. Taking into account $\phi$ is a polynomial of minimal degree such that $\phi u = 0$ then $\phi = A E^\nu$ divides $E^{\nu-1}(A'E - AF_0)$. So, $E$ divides $AF_0$. But $E$ and $F_0$ are coprime then $E$ divides $A$, a contradiction. Thus, $\phi(x) = E^k(x)$.

From (2.5) and from (2.1), we get

$$E^{k-1}F_ku = 0.$$ 

Since $E$ and $F_k$ are coprime, there exist two polynomials $S_i$, $i = 1, 2$ such that

$$S_1(x)E(x) + S_2(x)F_k(x) = 1.$$ 

Then,

$$S_1(x)E^k(x) + S_2(x)E^{k-1}(x)F_k(x) = E^{k-1}(x).$$

Multiplying by $u$, we get $E^{k-1}u = 0$. This contradicts the fact that $\phi = E^k$ has minimal degree and satisfies $\phi u = 0$. Hence, the weak regularity of $u$ follows. \qed

A.2.1. $E$ and $F$ are coprime and $\mu(E, F) \geq 2$.

**Lemma 2.8.** Let $u \in \mathcal{F}^*$ satisfy (2.1), with pseudo-class $t \geq 1$, $E$ and $F$ coprime polynomials, and $\mu(E, F) \geq 2$. Then the following statements are equivalent.

i) $u$ is weakly-regular.

ii) $E^{\mu(E, F) - 1}u \neq 0$.

**Proof.** From the assumption, let consider the linear functional $v = E^{\mu(E, F) - 1}u$. From Property 1.5, i), $u$ is a weakly-regular linear functional if and only if $v \neq 0$ and $v$ is weakly-regular. From (2.1), when $v \neq 0$, it satisfies

$$(Ev)' + F_\mu(E, F)v = 0,$$

where $E$ and $F_\mu(E, F) - (m - 1)E' = F_{m + \mu(E, F) - 1}$ are coprime, $m \geq 1$. Since $E$ and $E' = F_\mu(E, F) = F_{\mu(E, F) - 1}$ are coprime, then the pseudo-class of $v$ is $t \geq 1$. Therefore, from Lemma 2.7 $v$ is weakly-regular. Hence, $u$ is weakly-regular if and only if $E^{\mu(E, F) - 1}u \neq 0$. \qed

A.2.2. $E$ and $F$ are coprime and $\mu(E, F) \geq 2$.

**Proposition 2.9.** Let $u \in \mathcal{F}^*$ be a linear functional such that (2.1) holds with pseudo-class $t \geq 1$, and $G$ be the greatest common divisor of $E$ and $F$, with $E = \tilde{E}$ and $F = \tilde{F}$. The following statements are equivalent.

i) $u$ is weakly-regular.

ii) (i). If $\deg \tilde{E} = 0$, then $\deg \tilde{F} \geq 1$ and $Gu \neq 0$.

(ii). If $\deg \tilde{E} \geq 1$, then $GE^{\mu(\tilde{E}, \tilde{F}) - 1}u \neq 0$.

**Proof.** Consider $v = GE^{\mu(\tilde{E}, \tilde{F}) - 1}u$. The linear functional $u$ is weakly-regular if and only if $v \neq 0$ and $v$ is weakly-regular. But, if $v \neq 0$, then

$$(\tilde{E}v)' + \tilde{F}_\mu(\tilde{E}, \tilde{F})v = 0,$$

where $\tilde{E}$ and $\tilde{F}_\mu(\tilde{E}, \tilde{F}) - (m - 1)\tilde{E}' = \tilde{F}_{m + \mu(\tilde{E}, \tilde{F}) - 1}$ are coprime, $m \geq 1$. Since $\tilde{E}$ and $\tilde{E}' + \tilde{F}_\mu(\tilde{E}, \tilde{F}) = \tilde{F}_{\mu(\tilde{E}, \tilde{F}) - 1}$ are coprime, then $\tilde{t} = \deg \tilde{E}$ is the pseudo-class of $v$. Two cases appear.

(i). $\tilde{t} = 0$. According to Lemma 2.6 the non-zero linear functional $v$ is weakly-regular if and only if $\deg \tilde{F} \geq 1$. In this case, $u$ is weakly-regular if and only if $Gu \neq 0$ and $\deg \tilde{F} \geq 1$.

(ii). $\tilde{t} \geq 1$. The non-zero linear functional $v$ is weakly-regular, from Lemmas 2.7 and 2.8. In this case, $u$ is weakly-regular if and only if $GE^{\mu(\tilde{E}, \tilde{F}) - 1}u \neq 0$. \qed

**Remark 2.10.** When the linear functional $u$ solution of (2.1) satisfies $(u)_0 \neq 0$, and is weakly-regular, then we must have $\deg F \geq 1$. If not, $F(x) = \lambda \in \mathbb{C}$, then $(Eu)' + \lambda u = 0$ holds. So, from $(Eu)' + \lambda u, 1 = 0$, we get $\lambda(u)_0 = 0$. Hence, $\lambda = 0$, and, as a consequence, $Eu = 0$. This contradicts the weak-regularity of $u$.

2.2. Weak-semiclassical and semiclassical functionals. Let introduce the following definitions.

**Definition 2.11.** The linear functional $u$ said to be a weak-semiclassical functional when it is weakly-regular and satisfies (2.1), where the pair $(E, F)$ is admissible.

Notice that every semicllinear functional $u$ is also regular [7]. A weak-semiclassical functional $u$ satisfies an infinity number of first-order linear differential equations: for $\chi \in \mathcal{F}$, $u$ also fulfills

$$(E_1u)' + F_1u = 0,$$

with $E_1(x) = \chi(x)E(x)$, and $F_1(x) = \chi(x)F(x) - \chi'(x)E(x)$. So, if $s = s(E, F) = \max(t - 2, p - 1)$ and taking into account the admissibility condition of the pair $(E, F)$, i.e. $\Delta_q(E, F) \neq 0$, then we get $s_1 = \ldots = s$.}
\( s(E_1, F_1) = s + q \). Hence, we can associate with the weak-semiclassical functional \( u \) a subset \( h(u) \) of nonnegative integers such that \( m \) belongs to \( h(u) \) if and only if \( m = s(E_2, F_2) \) where \( (E_2, F_2) \) is an admissible pair of polynomials satisfying (2.1).

**Definition 2.12.** The minimum element \( s \) of \( h(u) \) is said to be the class of \( u \). When \( s = 0 \), the weak-semiclassical (resp. semiclassical) functional is called weak-classical (resp. classical) functional.

**Lemma 2.13.** Let \( u \) be a weak-semiclassical functional such that

\[
(E_i u)' + F_i u = 0, \quad \text{with} \quad s_i = \max(t_i - 2, p_i - 1), \quad i = 1, 2.
\]

Let denote by \( E \) the greatest common divisor of \( E_1 \) and \( E_2 \). Then, there exists a polynomial \( F \) such that

\[
(E u)' + F u = 0,
\]

with \( s = \max(t - 2, p - 1) = s_i - t_i + t, \quad i = 1, 2, \quad \text{where} \quad t = \deg E \quad \text{and} \quad p = \deg F. \]

**Proof.** See in [8] Lemma 3.3 and replace regularity by weak-regularity.

**Proposition 2.14.** For each weak-semiclassical functional \( u \), the pair \( (E, F) \) that realizes the minimum of \( h(u) \) is unique.

**Proof.** See in [8] Proposition 3.4 and replace regularity by weak-regularity.

**Proposition 2.15.** The class of the weak-semiclassical functional \( u \) satisfying (2.1) is \( s \) if and only if

\[
\prod_c \left( | F(c) + E'(c) | + | \langle u, \theta_c F + \theta_c^2 E \rangle | \right) > 0,
\]

where \( c \) belongs to the set of zeros of \( E \).

**Proof.** See in [8] Proposition 3.5 and replace regularity by weak-regularity.

**Proposition 2.16.** Let \( u \) be a weak-semiclassical functional satisfying \( (E u)' + F u = 0 \), where \( E \) monic, \( t = \deg E, \quad p = \deg F \geq 1 \), and \( s = \max(t - 2, p - 1) \). The following statements are equivalent.

i) The pseudo-class of \( u \) is \( t \).

ii) The class of \( u \) is \( s \).

**Proof.** It is a straightforward consequence of Lemmas 2.3 and 2.13.

**Remark 2.17.** Let \( u \) be a weak-semiclassical functional satisfying (2.1), with \( \deg E \geq 1 \). For each zero \( c \) of \( E \) and an integer \( m \geq 1 \), let consider the following linear functional

\[
v(m, c) = (x - c)^{m-1} u.
\]

Obviously, \( v(m, c) \) is weakly-regular and satisfies

\[
(E v(m, c))' + \hat{F}_m v(m, c) = 0.
\]

From Lemma 2.9, the pair of polynomials \((E, \hat{F})\) is admissible and has associated a nonnegative integer number \( s \). Thus, there exists an integer number \( k \geq 1 \) such that \( \langle v(k, c) \rangle_0 \neq 0 \). Otherwise, one has \( \langle u, (x-c)^{m-1} \rangle = 0, \quad m \geq 1 \). Then \( u = 0 \), a contradiction.

### 3. Classical Case.

It is well known that if \( s = 0 \) and the linear functional \( u \) is regular then we recover the classical functionals (Hermite, Laguerre, Bessel, and Jacobi) [1, 9, 10]. By a shift we get the following canonical classical functionals

\[
\begin{align*}
C_1. & \quad E(x) = 1, \quad F(x) = 2x. \\
C_2. & \quad E(x) = x, \quad F(x) = x - \alpha - 1. \\
C_3. & \quad E(x) = x^2, \quad F(x) = -2(\alpha x + 1). \\
C_4. & \quad E(x) = x^2 - 1, \quad F(x) = -(\alpha + \beta + 2)x + \alpha - \beta.
\end{align*}
\]

The functional \( u \) is the Hermite functional denoted \( H \).

The functional \( u \) is the Laguerre functional denoted \( L(\alpha) \). It is regular if and only if \( \alpha \neq -n, \quad n \geq 1 \).

The functional \( u \) is the Bessel functional denoted \( B(\alpha) \). It is regular if and only if \( \alpha \neq -\frac{n}{2}, \quad n \geq 0 \).

The functional \( u \) is the Jacobi functional denoted \( J(\alpha, \beta) \). It is regular if and only if \( \alpha \neq -n, \beta \neq -n \), and \( \alpha + \beta \neq -n - 1, \quad n \geq 1 \).

Notice that the polynomials \((E, F)\) in the above four canonical classical cases, \( C_i, \quad i = 1, \ldots, 4 \), are coprime, \( m \geq 1 \).

In the theory of first-order linear differential equations, the weak-regularity of the functional could reach its regularity, what is true here. First, we need to show the invariance of the weak-semiclassical character by shifting.
Lemma 3.1. When \( u \) is a weak-semiclassical functional of class \( s \), satisfying (2.1), then \( \tilde{u} = (h_{a^{-1} \circ \tau_{-b}})u \) is also a weak-semiclassical functional with the same class \( s \). It satisfies
\[
(\tilde{E}\tilde{u})' + \tilde{F}\tilde{u} = 0,
\]
where \( \tilde{E}(x) = a^{-1}E(ax + b) \) and \( \tilde{F}(x) = a^{1-t}F(ax + b) \).

Proof. The weak-regularity of \( \tilde{u} \) and the admissibility conditions of the pair \( (\tilde{E}, \tilde{F}) \) follow from Property 1.5, b) and Lemma 1.10, respectively. Finally, for the functional equation and the class of \( \tilde{u} \), see [1]. \( \square \)

Proposition 3.2. Let \( u \) be a linear functional satisfying \((Eu)' + Fu = 0\), where \( E \) monic, \( \deg E \leq 2 \), \( \deg F = 1 \), and the pair of polynomials \((E, F)\) is admissible. The following statements are equivalent.

i) \( u \) is regular.

ii) For each integer \( m \geq 1 \), \( E \) and \( F_m \) are coprime.

iii) \( u \) is weakly-regular.

Proof. i) \( \Rightarrow \) ii) \( \Rightarrow \) iii). It is straightforward.

iii) \( \Rightarrow \) i). It is sufficient to show that \( v \) is regular. So the following four situations must be analyzed:

C1. \( \deg(E) = 0 \). We can write \( E(x) = 1 \) and \( F(x) = cx + d \), \( c \neq 0 \). The shifted functional \( v = (h_{a^{-1} \circ \tau_{-b}})u \), where \((a, b) \in \mathbb{C}^* \times \mathbb{C} \) such that \( a^2 = \frac{2}{c} \) and \( b = \frac{2}{c} \), satisfies
\[
v' + 2\alpha v = 0, \quad (v)_0 = 1.
\]
The Hermite functional is the unique solution of (3.1). Hence, \( u \) is regular as the shifted of a regular functional.

C2. \( \deg(E) = 1 \). We can write \( E(x) = x + \xi \) and \( F(x) = cx + d \), \( c \neq 0 \). Let \( v = (h_{a^{-1} \circ \tau_{-b}})u \), where \( a = \frac{1}{c} \), \( b = -\xi \) and \( \alpha = \frac{\xi}{c} - d - 1 \). The functional \( v \) satisfies
\[
(vx')' + (x - \alpha - 1)v = 0, \quad (v)_0 = 1.
\]
Applying (3.2) to \( x^n \), \( n \geq 0 \), we get
\[
(v)_{n+1} = [n - (\alpha + 1)](v)_n, \quad n \geq 0, \quad (v)_0 = 1.
\]
Notice that \( \alpha \neq -n, \ n \geq 1 \). Otherwise, there exists an integer \( n_0, n_0 \geq 0 \), such that \( \alpha = -n_0 - 1 \). From (3.3), one has \( x^{n_0+1}v = 0 \). This contradicts the weak-regularity of \( v \), as the shifted of a weakly-regular functional. Therefore, \( v \) is the Laguerre functional. Thus, \( u \) is regular.

C3. \( \deg(E) = 2 \) and \( E \) has a double zero. We can write \( E(x) = (x + \xi)^2 \) and \( F(x) = cx + d \). Since \( \deg(F) = 1 \) and taking into account \( (E, F) \) is an admissible pair we get \( c \neq n, \ n \geq 0 \). If \( \alpha = -(c/2) \), then \( \alpha \neq -(n/2), \ n \geq 0 \). Let \( a = \frac{c\xi - d}{2} \). Then, \( a \neq 0 \). Otherwise, the functional \( v = \tau_{\xi}u \) satisfies
\[
(x^2v')' - 2\alpha xv = 0,
\]
and applying (3.4) to \( x^n \), \( n \geq 0 \), we get \( (n+2\alpha)(v)_{n+1} = 0, \ n \geq 0 \). Since \( \alpha \neq -(n/2), \ n \geq 0 \), then \( xv = 0 \), and this leads to a contradiction. So, it is possible to consider the functional \( v = (h_{a^{-1} \circ \tau_{-b}})u \), where \( a = \frac{c\xi - d}{2} \) and \( b = -\xi \). The shifted functional \( v \) satisfies
\[
(x^2v')' - 2\alpha x + 1)v = 0, \quad (v)_0 = 1,
\]
where \( \alpha \neq -(n/2), \ n \geq 0 \). Thus, \( v \) is the Bessel functional and \( u \) is regular.

C4. \( \deg(E) = 2 \) and \( E \) has two different zeros. We can write \( E(x) = (x + \xi_1)(x + \xi_2) \), with \( \xi_1 \neq \xi_2 \), and \( F(x) = cx + d \), where \( c \neq 0, \ n \geq 0 \). Let \( v = (h_{a^{-1} \circ \tau_{-b}})u \), where \( a = \frac{\xi_2 - \xi_1}{2} \) and \( b = \frac{\xi_1 + \xi_2}{2} \). We take \( \alpha = \frac{c(b-a) + d - 2a}{2a} \) and \( \beta = -\frac{c(a + b) + d + 2a}{2a} \). The shifted functional \( v \) satisfies
\[
((x^2-1)v')' - (\alpha + \beta + 2)x + \alpha - \beta)v = 0, \quad (v)_0 = 1
\]
with \( \alpha + \beta = -c - 2 \neq -n - 2, \ n \geq 0 \). Applying (3.6) to \( (x - 1)^n \), \( n \geq 0 \), we get
\[
(n + \alpha + \beta + 2)v_{1,n+1} = -2(n + \beta + 1)v_{1,n}, \quad n \geq 0.
\]
On the other hand, applying (3.6) to \( (x + 1)^n \), \( n \geq 0 \), we get
\[
(n + \alpha + \beta + 2)v_{-1,n+1} = 2(n + \alpha + 1)v_{-1,n}, \quad n \geq 0.
\]
Suppose there exists an integer \( n_0, n_0 \geq 1 \), such that \( \beta = -n_0 - 1 \) (resp. \( \alpha = -n_0 - 1 \)). Since \( \alpha + \beta \neq -n - 2, \ n \geq 0 \), from (3.7), (resp. (3.8)), then \( (x - 1)^{n_0 + 1}v = 0 \), (resp. \( (x + 1)^{n_0 + 1}v = 0 \)). This contradicts the weak-regularity of \( v \).

As a consequence, \( \alpha + \beta \neq -n, \ n \geq 2, \alpha \neq -n, \ n \geq 1, \) and \( \beta \neq -n, \ n \geq 1 \). The functional \( v \) is the Jacobi functional, then \( u \) is regular. \( \square \)

Proposition 3.3. Let \( \{C_n\}_{n \geq 0} \) be a sequence of monic polynomials with dual sequence \( \{c_n\}_{n \geq 0} \), such that \( E(x)C_n(x) + F(x)C'_{n+1}(x) = \lambda_{n+1}C_{n+1}(x) \), \( n \geq 0 \), where \( E \) monic, \( \deg E \leq 2 \), \( \deg F = 1 \), and the pair of polynomials \((E, F)\) is admissible. The following statements are equivalent.

i) \( \{C_n\}_{n \geq 0} \) is orthogonal with respect to \( c_0 \).
ii) For each integer $m \geq 1$, $E$ and $F_m$ are coprime.

iii) $c_0$ is weakly-regular.

\textbf{Proof.} From the higher degree coefficients in the second-order differential equation, and the admissibility condition of the pair $(E, F)$, we get

$$\lambda_{n+1} = (n+1)\left(\frac{n+1}{2} - F'(0)\right) \neq 0, \ n \geq 0. \quad (3.9)$$

On the other hand, we get

$$\langle E c_0 \rangle' + F c_0 = 0. \quad (3.10)$$

i) $\Rightarrow$ ii). It is a consequence of (3.10), the regularity of $c_0$, and Proposition 3.2.

ii) $\Rightarrow$ iii). It follows from Proposition 3.2.

iii) $\Rightarrow$ i). From Proposition 3.2, the linear functional $c_0$ is regular. Thus, the sequence $\{C_n\}_{n \geq 0}$ will be orthogonal with respect to $c_0$, see in [8] Proposition 2.9.

4. Applications.

\textbf{A.1.} Assume that $\nu$ is a classical functional. Let $u$ be a regular functional such that

$$Au = \lambda Bu. \quad (4.1)$$

Here $\lambda \in \mathbb{C}^*$ and $A, B$ are two monic polynomials. This kind of perturbations have been analyzed in [11]. The linear functional $u$ is semi-classical. Indeed, if we assume that the functional $\nu$ satisfies $E \nu' + F \nu = 0$, where $E$ monic, $\deg E \leq 2$, $\deg F = 1$, and the pair $(E, F)$ is admissible, then it is easy to prove that $u$ satisfies

$$\langle ABu \rangle' + A(BF - 2B' E)u = 0. \quad (4.2)$$

From Proposition 3.2, we can characterize in a natural way the MOPS with respect to $u$. Indeed

\textbf{Proposition 4.1.} Let $B$ be a monic polynomial, $\deg B = t$, and $\{B_n\}_{n \geq 0}$ be the sequence of monic orthogonal polynomials with respect to $u$. The following statements are equivalent.

i) There exist a monic polynomial $A$, a non-zero-constant $\lambda$, and a classical functional $\nu$ such that

$$Au = \lambda Bu. \quad (4.3)$$

ii) There exist an integer $s \geq 0$, an MPS $\{\Pi_{n+t}\}_{n \geq s}$, deg $\Pi_{n+t} = n + t$, $n \geq s$, and non-zero-constants $\vartheta_n$, $n \geq s$, such that

$$\vartheta_n B(x) B_{n+1}(x) = E(x) \Pi_{n+t}'(x) - F(x) \Pi_{n+t}(x), \quad n \geq s, \quad (4.4)$$

where $(E, F)$ is an admissible pair of polynomials, $E$ monic, $\deg E \leq 2$, $\deg F = 1$, and $E$ and $F_m$ are coprime, $m \geq 1$.

\textbf{Proof.} i) $\Rightarrow$ ii). Let $\{C_n\}_{n \geq 0}$ be the MOPS with respect to the functional $\nu$. From Lemma 2.1, if $s = deg A$, then

$$B(x) B_{n+1}(x) = \sum_{\nu = n-s}^{n+t} \frac{\langle u, AC_{\nu+1} B_{n+1} \rangle}{\lambda(\nu, C^2_{\nu+1})} C_{\nu+1}(x), \quad n \geq s. \quad (4.5)$$

On the other hand the classical sequence $\{C_n\}_{n \geq 0}$ satisfies a second-order differential equation [3]

$$E(x) C_{\nu+1}'(x) - F(x) C_{\nu+1}'(x) = \lambda_{\nu+1} C_{\nu+1}(x), \quad \nu \geq 0, \quad (4.6)$$

where $\lambda_{\nu+1} = (\nu + 1)(\nu + 2) - F'(0) \neq 0$, $\nu \geq 0$. Using (4.6), from (4.5) we deduce (4.4), with

$$\vartheta_n = \frac{\lambda_{n+1+t}}{n + t + 1}, \quad n \geq s, \quad (4.7)$$

$$\Pi_{n+t}(x) = \sum_{\nu = n-s}^{n+t} \frac{\lambda_{\nu+1+t}}{\lambda_{\nu+1}} \frac{\langle u, AC_{\nu+1} B_{n+1} \rangle}{\lambda(\nu, C^2_{\nu+1})} C_{\nu+1}'(x), \quad \nu \geq s. \quad (4.8)$$

for $n \geq s$.

ii) $\Rightarrow$ i). From the assumption ii) and Proposition 3.2, let consider the classical functional $\nu$ satisfying $(E \nu)' + F \nu = 0$. From (4.4), we get $\langle B u, B_{n+1} \rangle = 0$, $n \geq s$. Thus, there exists an integer $r$, $0 \leq r \leq s$, such that $\langle B u, B_r \rangle \neq 0$. Otherwise, since $\langle B u, B_n \rangle = 0$, $n \geq 0$, then $B u = 0$. This contradicts the regularity of $u$. As a consequence, $\langle B u, B_{n+1} \rangle = 0$, $n \geq s$, and $\langle B u, B_r \rangle = 0$.

From Lemma 2.1, we get $B u = \sum_{\nu = 0}^{s} \langle B u, B_{\nu} \rangle u_{\nu}$, and by using (1.4), we finally obtain (4.3), with

$$\lambda = \frac{\langle u, B^2_r \rangle}{\langle B u, B_r \rangle},$$

$$A(x) = \sum_{\nu = 0}^{s} \frac{\langle B u, B_{\nu} \rangle}{\langle B u, B_r \rangle} \langle u, B^2_{\nu} \rangle B_{\nu}(x). \quad \Box$$

\textbf{A.2.} For each fixed $\mu \in \mathbb{C}^*$, let $u(\mu)$ be the linear functional satisfying

$$(E u(\mu))' + F u(\mu) = 0, \quad (u(\mu))_0 = 1, \quad (u(\mu))_1 = 0, \quad (4.9)$$

with $E(x) = x$ and $F(x) = 2x^2 - (2\mu + 1)$. If $(u(\mu))_n$, $n \geq 0$, denote the moments of $u(\mu)$, we get

$$(u(\mu))_{n+2} = \frac{(n + 2\mu + 1)}{2} (u(\mu))_n, \quad n \geq 0, \quad (u(\mu))_1 = 0, \quad (u(\mu))_0 = 1. \quad (4.10)$$
Clearly $u(\mu)$ is a symmetric linear functional.

Proposition 4.2. For each fixed $\mu \in C^*$, let $u(\mu)$ be the linear functional satisfying (4.9). The following statements are equivalent.

i) $u(\mu)$ is regular.

ii) $u(\mu)$ is weakly-regular.

iii) The linear functional $\sigma(u(\mu))$ is weakly-regular.

Proof. i) $\Rightarrow$ ii). The regularity of $u(\mu)$ yields weak-regularity.

ii) $\Rightarrow$ iii). According to Proposition 1.7 and taking into account that $u(\mu)$ is symmetric and weakly-regular, we deduce that $\sigma(u(\mu))$ is weakly-regular.

iii) $\Rightarrow$ i). From (4.9) the linear functional $\sigma(u(\mu))$ satisfies

\[(x\sigma(u))' + (x - \alpha - 1)\sigma(u) = 0, \alpha = \mu - \frac{1}{2}. \quad (4.11)\]

From Proposition 3.2, and taking into account the weak-regularity of $\sigma(u)$ and the admissibility condition of the pair $(x, x - \alpha - 1)$, the regularity of $\sigma(u)$, i.e., $\alpha \neq -n, n \geq 1$ follows. Therefore, $\mu \neq -n - (1/2), n \geq 0$. Thus, $u(\mu)$ will be a semiclassical linear functional of class one. More precisely, it is the generalized Hermite functional denoted $H(\mu)$ and $\sigma(u)$ is the Laguerre linear functional.

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